Exercise 6B: Solutions

1 a $P: 1 > 0$ (true)
not $P: 1 \leq 0$ (false)

b $P: 4$ is divisible by 8 (false)
not $P: 4$ is not divisible by 8 (true)

c P : Each pair of primes has an even sum (false)
not P : Some pair of primes does not have an even sum (true)

d P : Some rectangle has 4 sides of equal length (true)
not P : No rectangle has 4 sides of equal length (false)

2 a $P: 14$ is divisible by 7 and 2 (true)
not $P: 14$ is not divisible by 7 or 14 is not divisible by 2 (false)

b $P: 12$ is divisible by 3 or 4 (true)
not $P: 12$ is not divisible by 4 and 12 is not divisible by 3 (false)

c $P: 15$ is divisible by 3 and 6 (false)
not $P: 15$ is not divisible by 3 or 15 is not divisible by 6 (true)

d $P: 10$ is divisible by 2 or 5 (true)
not $P: 10$ is not divisible by 2 or 10 is not divisible by 5 (false)

3 We will prove that Alice is a knave, and Bob is a knave.

Suppose Alice is a knight
$\Rightarrow$ Alice is telling the truth
$\Rightarrow$ Alice is a knave
$\Rightarrow$ Alice is a knight and a knave
This is impossible.
$\Rightarrow$ Alice is a knave
$\Rightarrow$ Alice is not telling the truth
$\Rightarrow$ Alice is a knight OR Bob is a knave
$\Rightarrow$ Bob is a knave, as Alice is not a knight
$\Rightarrow$ Alice and Bob are both knaves.

4 a If there are no clouds in the sky, then it is not raining.

b If you are not happy, then you are not smiling.

c If $2x \neq 2$, then $x \neq 1$.

d If $x^5 \leq y^5$, then $x \leq y$.

e Option 1: If n is not odd, then n^2 is not odd.
Option 2: If n is even, then n^2 is even.

f Option 1: If mn is not odd, then n is not odd or m is not odd.
Option 2: If If mn is even, then n is even or m is even.

g Option 1: If n and n are not both even or both odd, then $m + n$ is not even.
Option 2: If n and n are not both even or both odd, then $m + n$ is odd.

5 a Contrapositive: If n is even then $3n + 5$ is odd.
Proof: Suppose n is even. Then $n = 2k$, for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned}3n + 5 &= 3(2k) + 5 \\ &= 6k + 5 \\ &= 6k + 4 + 1 \\ &= 2(3k + 2) + 1\end{aligned}$$

is odd.

b Contrapositive: If n is even, then n^2 is even.
Proof: Suppose n is even. Then $n = 2k$, for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} n^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2(2k^2) \end{aligned}$$

is even.

c Contrapositive: If n is even, then $n^2 - 8n + 3$ is odd.
Proof: Suppose n is even. Then $n = 2k$, for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} n^2 - 8n + 3 &= (2k)^2 - 8(2k) + 3 \\ &= 4k^2 - 16k + 3 \\ &= 4k^2 - 16k + 2 + 1 \\ &= 2(2k^2 - 8k + 1) + 1 \end{aligned}$$

is odd.

d Contrapositive: If n is divisible by 3, then n^2 is divisible by 3.
Proof: Suppose n is divisible by 3. Then $n = 3k$, for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} n^2 &= (3k)^2 \\ &= 9k^2 \\ &= 3(3k^2) \end{aligned}$$

is divisible by 3.

e Contrapositive: If n is even, then $n^3 + 1$ is odd.
Proof: Suppose n is even. Then $n = 2k$, for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} n^3 + 1 &= (2k)^3 + 1 \\ &= 8k^3 + 1 \\ &= 2(4k^3) + 1 \end{aligned}$$

is odd.

f Contrapositive: If m or n are divisible by 3, then mn is divisible by 3.
Proof: If m or n is divisible by 3 then we can assume that m is divisible by 3. Then, $m = 3k$, for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} mn &= (3k)n \\ &= 3(kn) \end{aligned}$$

is divisible by 3.

g Contrapositive: If $m = n$, then $m + n$ is even.
Proof: Suppose that $m = n$. Then

$$\begin{aligned} m + n &= n + n \\ &= 2n \end{aligned}$$

is even.

6 a Contrapositive: If $x \geq 0$, then $x^2 + 3x \geq 0$.
Proof: Suppose that $x \geq 0$. Then,

$$x^2 + 3x = x(x+3) \geq 0,$$

since $x \geq 0$ and $x + 3 \geq 0$.

b Contrapositive: If $x \leq -1$, then $x^3 - x \leq 0$.
Proof: Suppose that $x \leq -1$. Then,

$$x^3 - x = x^2(x-1) \leq 0,$$

since $x^2 \geq 0$ and $x - 1 \geq 0$.

c Contrapositive: If $x < 1$ and $y < 1$, then $x + y < 2$.
Proof: If $x < 1$ and $y < 1$ then,

$$x + y < 1 + 1 = 2,$$

as required.

d Contrapositive: If $x < 3$ and $y < 2$, then $2x + 3y < 12$.
Proof: If $x < 3$ and $y < 2$ then,

$$2x + 3y < 2 \times 3 + 3 \times 2 = 6 + 6 = 12,$$

as required.

7 a Contrapositive: If m is odd or n is odd, then mn is odd or $m + n$ is odd.

b Proof:
(Case 1) Suppose m is odd and n is odd. Then clearly mn is odd.
(Case 2) Suppose m is odd and n is even. Then clearly $m + n$ will be odd. It is likewise, if m is even and n is odd.

8 a We rationalise the right hand side to give,

$$\begin{aligned}\frac{x-y}{\sqrt{x}+\sqrt{y}} &= \frac{x-y}{\sqrt{x}+\sqrt{y}}\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}-\sqrt{y}} \\ &= \frac{(x-y)\left(\sqrt{x}-\sqrt{y}\right)}{\left(\sqrt{x}+\sqrt{y}\right)\left(\sqrt{x}-\sqrt{y}\right)} \\ &= \frac{(x-y)\left(\sqrt{x}-\sqrt{y}\right)}{(x-y)} \\ &= \sqrt{x}-\sqrt{y}.\end{aligned}$$

b If $x > y$ then $x - y > 0$. Then, using the above equality, we see that,

$$\sqrt{x} - \sqrt{y} = \frac{x-y}{\sqrt{x}+\sqrt{y}} > 0,$$

since the numerator and denominator are both positive. Therefore, $\sqrt{x} > \sqrt{y}$.

c Contrapositive: If $\sqrt{x} \leq \sqrt{y}$, then $x \leq y$.
Proof: If $\sqrt{x} \leq \sqrt{y}$ then, since both sides are positive, we can square both sides to give $x \leq y$.